Exercise 24

Use a calculator to evaluate the line integral correct to four decimal places.

$$\int_C \mathbf{F}\cdot d\mathbf{r}, \quad \text{where } \mathbf{F}(x,y,z) = yze^x\,\mathbf{i} + zxe^y\,\mathbf{j} + xye^z\,\mathbf{k}$$
$$\text{and } \mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}, \quad 0 \le t \le \pi/4$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/4} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\
&= \int_0^{\pi/4} \left\langle y(t)z(t)e^{x(t)}, z(t)x(t)e^{y(t)}, x(t)y(t)e^{z(t)}\right\rangle \cdot \frac{d}{dt}\langle \sin t, \cos t, \tan t\rangle\,dt \\
&= \int_0^{\pi/4} \left\langle (\cos t\tan t)e^{\sin t}, (\tan t\sin t)e^{\cos t}, (\sin t\cos t)e^{\tan t}\right\rangle \cdot \langle \cos t, -\sin t, \sec^2 t\rangle\,dt \\
&= \int_0^{\pi/4} \left[(\cos^2 t\tan t)e^{\sin t} + (-\tan t\sin^2 t)e^{\cos t} + (\sin t\cos t\sec^2 t)e^{\tan t}\right] dt \\
&= \int_0^{\pi/4} \left[(\sin t\cos t)e^{\sin t} - (\tan t\sin^2 t)e^{\cos t} + (\tan t)e^{\tan t}\right] dt.
\end{aligned}$$

Let $f(t)$ represent the integrand.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/4} f(t)\,dt$$

Use Simpson's rule with $n = 10$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &\approx \frac{\Delta t}{3}[f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_4) + 4f(t_5) \\
&\qquad + 2f(t_6) + 4f(t_7) + 2f(t_8) + 4f(t_9) + f(t_{10})] \\
&\approx \frac{\frac{\pi}{4}-0}{3(10)}\left[f(0) + 4f\left(\frac{\pi}{40}\right) + 2f\left(\frac{\pi}{20}\right) + 4f\left(\frac{3\pi}{40}\right) + 2f\left(\frac{\pi}{10}\right) + 4f\left(\frac{\pi}{8}\right)\right. \\
&\qquad \left. + 2f\left(\frac{3\pi}{20}\right) + 4f\left(\frac{7\pi}{40}\right) + 2f\left(\frac{\pi}{5}\right) + 4f\left(\frac{9\pi}{40}\right) + f\left(\frac{\pi}{4}\right)\right] \\
&\approx \frac{\frac{\pi}{4}-0}{3(10)}(32.5732) \\
&\approx 0.852766
\end{aligned}$$